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PROCEEDINGS
OF THE
NATIONAL ACADEMY OF SCIENCES

Volume 8 MARCH 15, 1922 Number 3

ON THE RELATION OF A CONTINUOUS CURVE TO ITS COMPLEMENTARY DOMAINS IN SPACE OF THREE DIMENSIONS[1]

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Communicated by E. H. Moore, February 1, 1922

Schoenflies has shown[2] that, in order that a closed, bounded and connected point-set, lying in a *plane* S, should be a continuous curve, it is necessary and sufficient that (1) if R is a domain complementary to M, every point of the boundary of R should be "accessible from all sides" with respect to R, and (2) if ϵ is any preassigned positive number, there do not exist infinitely many distinct domains, complementary[3] to M, and all of diameter[4] greater than ϵ. In the present paper I will exhibit examples of continuous curves, in *space of three dimensions*, which satisfy neither of these conditions.

If, in a *plane* S, a closed and bounded point-set K separates S into just two domains, S_1 and S_2, such that every point of K is a limit point both of S_1 and of S_2, then, in order that K should be a simple closed curve, it is[5] necessary and sufficient that every point of K should be accessible[6] from every point which does not belong to K. In this paper it will be shown that, for a space S of *three dimensions*, this condition is neither necessary nor sufficient in order that K should be a simple closed surface or in order that it should be a continuous curve. It will be shown, however, that a closed, bounded and connected point-set in space of three dimensions is a continuous curve provided it is the common boundary of two mutually exclusive domains both of which are *uniformly connected im kleinen.*[7]

1. *There exists, in a three dimensional space S, a continuous curve M which divides S into just two domains, S_1 and S_2, such that (a) the boundary of S_1 contains a point which is not accessible from any point of S_1, (b) the domain S_2 is uniformly connected im kleinen, (c) the boundary of S_1 is not a continuous curve.*

Example.—Let K denote a cube placed so that b, one of its bases, is horizontal. Let l and f denote two opposite lateral faces of K. Let

$ABCD$ and $EFGH$ denote two squares concentric with l and lying within it and such that $EFGH$ is within $ABCD$, AB and EF being horizontal. Let I and J denote points in the face f such that the planes IAB and JCD are parallel to b and such that I is equidistant from A and B, while J is equidistant from C and D. We thus have a right prism whose triangular bases are ABI and DCJ. Let P denote this prism and let Q denote the right pyramid whose base is the square $EFGH$ and whose apex is at O, a point half-way between I and J. For each positive integer n, let α_n denote a plane parallel to f and at a distance from f equal to a/n, where a is one-half the length of a side of K. For each n, let β_n denote a plane parallel to, and half-way between the planes α_n and α_{n+1}. Let I_n denote that portion of the interior of P which is intercepted between the planes α_n and β_n. Let R denote the interior of the pyramid Q. Let S_1 denote the point-set composed of R together with the sets $I_1, I_2, I_3, I_4, \ldots$. Let N denote the point-set composed of the surface of K together with its interior and let M denote N-R. The set M is a continuous curve, S_1 is one of its complementary domains, and the straight line interval JI is a portion of the boundary of S_1. But no point of JI, except O, is accessible from any point of S_1. The boundary of S_1 is not connected *im kleinen* at any point of JI except O. Clearly S_2, the exterior of the cube K, is uniformly connected *im kleinen.*

2. *If* ϵ *is any preassigned positive number, there exists a continuous curve M such that there are infinitely many distinct domains, complementary to M and all of diameter greater than* ϵ.

Example.—Consider again the cube K and the domains $I_1, I_2, I_3, \ldots$ of Example 1, it being understood that the distance between the planes IAB and JCD of that example is 2ϵ. Let M denote the point-set $N-(I_1+I_2+I_3+\ldots)$ where N has the same meaning as in Example 1. Here M is a continuous curve and each I_n is one of its complementary domains.

3. *There exists, in three dimensional space, a continuous curve M which divides space into just two domains,* S_1 *and* S_2, *such that (a) every point of M is a limit point both of* S_1 *and of* S_2, *(b) not every point of M is accessible from a point of* S_2.

Example.—Let K denote a square whose vertices A, B, C, D are the points $(0, -1, -1)$, $(0, 1, -1)$, $(0, 1, 1)$ and $(0, -1, 1)$, respectively. For each integer n (positive or negative) let K_n denote a square whose vertices, A_n, B_n, C_n, D_n, are the points $(1/n, -1, -1)$, $(1/n, 1, -1)$, $(1/n, 1, 1)$ and $(1/n, -1, 1)$, respectively. For each n let $\overline{K}_n$ denote a square whose vertices, $\overline{A}_n, \overline{B}_n, \overline{C}_n, \overline{D}_n$, are so situated that $\overline{A}_n$ is half-way between A_n and A_{n_1}, $\overline{B}_n$ is half-way between B_n and B_{n_1}, $\overline{C}_n$ is half-way between C_n and C_{n_1} and $\overline{D}_n$ is half-way between D_n and D_{n_1}, where n_1 denotes $n+1$ or $n-1$ according as n is positive or negative. For every integer n and every pair of integers i and j which are numerically less than $4^{|n|}$, let $\overline{P}_{ijn}$ denote that point, within the square $\overline{K}_n$, whose y and z are $i/4^{|n|}$ and $j/4^{|n|}$, respectively. For each n, let T_n denote the set of all points $[\overline{P}_{ijn}]$ for which i and j are even numbers numerically less than $4^{|n|}$ and let U_n denote the set of all points $[\overline{P}_{ijn}]$ such that i and j are odd numbers numerically less than $4^{|n|}$. For every point $\overline{P}_{ijn}$ belonging to T_n or to U_n let $\overline{K}_{ijn}$ denote a circle lying in the plane $\overline{K}_n$, with center at $\overline{P}_{ijn}$ and with a radius equal to $1/4^{|n|+2}$. Thus with each point of the set $T_n + U_n$ there is associated a definite circle having that point as center, and every one of these circles lies wholly without every other one. Now let P'_{ijn} and K'_{ijn} denote the orthogonal projections of P_{ijn} and $\overline{K}_{ijn}$, respectively, onto the plane of the square K_n, and let P_{ijn} denote the orthogonal projection of $\overline{P}_{ijn}$ onto the plane of K_n. For each point $\overline{P}_{ijn}$ belonging to the set T_n let S_{ijn} denote the right circular cylinder whose bases have the circles $\overline{K}_{ijn}$ and K'_{ijn} as their perimeters, and for each point $\overline{P}_{ijn}$ belonging to U_n let C_{ijn} denote the right circular cone whose base has $\overline{K}_{ijn}$ as its perimeter and whose apex is the point P_{ijn}. It is to be noted that S_{ijn} exists only when i and j are even and that C_{ijn} exists only when i and j are odd. For each n, let P_n denote the right parallelopiped which has K_n and $\overline{K}_n$ as opposite lateral faces and let R_n denote the interior of P_n. Let I_n denote that portion of R_n which remains after the removal of every point of R_n which lies either within, or on the surface of any C_{ijn} (i and j odd). For each n let L_n denote the point-set obtained by adding to I_n the interiors of all the cylinders S_{ijn} (i and j even) together with the interiors of the bases of these cylinders. Let K^* denote a rectangle lying in the plane $z=2$ and with sides equal and parallel to the respective sides of the upper base of P_1. Let P' denote a parallelopiped whose upper base is K^* and whose lower base is the upper base of P_1. Let P'' denote a parallelopiped whose upper base is K^* and whose lower base is the upper base of P_{-1}. Let L^* denote the point-set composed of the interiors of P' and P'' and the interiors of the upper bases of P_1 and P_{-1}. Let H denote the point-set $L^* + L_1 + L_2 + L_3 + \ldots$, and let M denote its boundary. The point-set H is a domain and M is a continuous curve which separates space into two domains S_1 and S_2 (where S_1 is H). Every point within the rectangle K belongs to M. But no point within this rectangle is accessible from any point in S_2, though every point of M is a limit point both of S_1 and of S_2. Neither S_1 nor S_2 is uniformly connected *im kleinen*.

4. *There exists, in a space S of three dimensions, a closed, connected and bounded point-set M which divides S into just two domains such that (a) both of these domains are uniformly connected im kleinen, and every point of M is accessible from one of these domains, but (b) M is not a continuous curve.*

Example.—Let K denote the cube which is bounded by the planes $x=0$,

$x=2, y=0, y=2, z=0, z=2$. For each positive integer n let A_n denote the point $(1/n, 1/n, 0)$ and let B_n denote the point $(1/n, 1/n, 2)$. Let M denote the point-set composed of the surface of K together with all the straight line intervals $A_1 B_1, A_2 B_2, A_3 B_3, \ldots$.

5. *There exists, in three dimensional space, a closed, bounded and connected point-set M which is not a continuous curve but which divides space into just two domains S_1 and S_2 such that (a) every point of M is accessible both from every point of S_1 and from every point of S_2, (b) the domain S_1* is uniformly connected *im kleinen.*

Example.—Returning to Example 3, let H denote a cube whose upper base lies in the plane $z=1$, whose center is at the point $(0, 0, -1)$, and one of whose faces is in the plane $x=2$. For every positive integer n, the upper base of the parallelopiped P_n of Example 3 is a part of the upper base of H but the remainder of P_n is wholly within H. Let R denote the set of points composed of (*a*) the exterior of H, (*b*) the interiors of the upper bases of the parallelopiped $P_1, P_2, P_3, \ldots$, (*c*) the interiors of $P_1, P_2, P_3, \ldots$ and (*d*) the interiors of those cylinders S_{ijn} (of Example 3) for which n is positive, together with the interiors of the bases of those cylinders. Let M denote the boundary of R. The point-set M is closed, connected and bounded and it divides space into just two domains, S_1 and S_2 (where S_1 is R), and furthermore every point of M is accessible from every point of S_1 and from every point of S_2. But M is not a continuous curve. It fails to be connected *im kleinen* at the point $(0, 0, 0)$. The domain S_1 is uniformly connected *im kleinen* but the domain S_2 is not.

THEOREM 1. *If, in a three dimensional space S, the closed, bounded and connected point-set M divides S into just two domains S_1 and S_2 such that (a) every point of M is a limit point both of S_1 and of S_2, and (b) both S_1 and S_2 are uniformly connected im kleinen; then M is a continuous curve.*

This theorem will be proved with the assistance of the following lemma. A proof of this lemma will be given elsewhere.

Lemma. *Suppose that, in space of three dimensions, (1) M is a closed point-set, (2) $A_1 B_1$ and $A_2 B_2$ are two simple continuous arcs each of which consists of a finite number of straight line intervals, (3) neither of the arcs $A_1 B_1$ and $A_2 B_2$ has a point in common with M, (4) every simple continuous arc which joins a point of $A_1 B_1$ to a point of $A_2 B_2$ contains at least one point of M. Then M contains a closed and connected point-set which contains at least one point in common with each of the straight line intervals $A_1 A_2$ and $B_1 B_2$ and has a diameter less than, or equal to, twice that of the point-set composed of the two arcs $A_1 B_1$ and $B_1 B_2$.*

Proof of Theorem 1.—Suppose, on the contrary, that M is not connected *im kleinen*. Then there exists a positive number ϵ and two infinite sequences of points $A_1, A_2, A_3, \ldots$ and $B_1, B_2, B_3, \ldots$, belonging to M, such that $\lim_{n=\infty} A_n B_n = O$ and such that, for no n, do A_n and B_n lie together in a closed and connected subset of M of diameter less than or equal to ϵ. But, since the domains S_1 and S_2 are uniformly connected *im kleinen*, there exists a positive number $\delta\epsilon$ such that if X and Y are two points both belonging to S_1, or both belonging to S_2, and such that $XY < \delta\epsilon$ then X and Y can be joined by a simple broken line of diameter less than $\epsilon/9$ which lies wholly in S_1 or wholly in S_2. There exists an integer m such that A_m, $B_m < \delta\epsilon/3$. For each positive integer n there exists spheres α_n and β_n, with centers at A_m and B_m, respectively, and with diameters less than each of the numbers $\delta\epsilon/3$, $\epsilon/9$ and $1/n$. There exist, within α_n, points E_n and E'_n belonging to S_1 and S_2, respectively, and within β_n, points D_n and D'_n belonging to S_1 and S_2, respectively. There exist broken lines $D_n\ E_n$ and $D'_n\ E'_n$ lying in S_1 and S_2, respectively, and with diameters less than $\epsilon/9$. Every simple continuous arc from a point of $D_n\ E_n$ to a point of $D'_n\ E'_n$ contains a point of M. It follows from the above lemma that M contains a closed and connected point-set M_n, of diameter less than or equal to $2\epsilon/3$, which contains a point within α_n and a point within β_n. The sequence of closed and connected point-sets $M_1, M_2, M_3, \ldots$ has, as its limiting set,[8] a closed and connected subset of M which contains both A_m and B_m and has a diameter less than ϵ. Thus the supposition that M is not connected *im kleinen* has led to a contradiction.

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[1] Presented, in part, to The American Mathematical Society, October 29, 1921.

[2] Schoenflies, A., *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten*, Zweiter Teil, Leipsig, 1908, p. 237.

[3] A connected set of points M lying in a space S of one or more dimensions is said to be a *domain* (with respect to S) provided there exists, in S, a closed (or vacuous) point-set N such that $M=S-N$. The domain R is said to be complementary to the closed point-set M if (1) R contains no point of M, and (2) the boundary of R is a subset of M.

[4] The diameter of a bounded point-set M is the smallest number d such that if X and Y are any two points of M then the distance from X to Y is less than d.

[5] Cf. Schoenflies, A., "Ueber einen grundlegenden Satz der Analysis Situs," *Göttingen, Nachr. Ges. Wiss.*, 1902 (185).

[6] A point X, belonging to a closed point-set K, is said to be *accessible* from a point Y which does not belong to K if there exists a simple continuous arc from X to Y which has no point except X in common with K.

[7] A point-set M is said to be *connected "im kleinen"* if, for every point P of M and every positive number e, there exists a positive number δ_{ep} such that if X is a point of M at a distance less than δ_{ep} from the point P, then X and P lie together in a connected subset of M every point of which is at a distance of less than e from the point P. The set M is said to be *uniformly* connected *im kleinen* if for every positive number e there exists a positive number δ_e such that if P_1 and P_2 are two points of M at a distance apart less than δ_e then they lie together in a connected subset of M every point of which is at a distance of less than e from P_1. Cf. Hahn, H., *Wien. Ber.*, **123**, 1914 (2433). Hahn shows that in order that a closed, bounded and connected point-set M should be a continuous curve it is necessary and sufficient that M should be connected *im kleinen*. See also, however, Mazurkiewiez, S., *Fundamenta Mathematicae*, **1**, 1920 (166–209). In this article the author refers to earlier articles by himself, published, in 1913 and 1916, in a journal (*C. R. Soc. Sc. Varsovie*) to which I do not, at present, have access.

[8] The point-set t is said to be the limiting set of the sequence of point-sets t_1, t_2, t_3, ... provided that (*a*) each point of t is the sequential limit point of an infinite subsequence of some sequence of points, P_1, P_2, P_3, ... sucht hat, for every n, P_n belongs to t_n, and (*b*) if P_1, P_2, P_3, ... is a sequence of points such that, for every n, P_n belongs to t_n then t contains the sequential limit point of every subsequence of P_1, P_2, P_3, ... which has a sequential limit point.

CHARACTERISTIC EFFECTS UPON GROWTH, OESTRUS AND OVULATION INDUCED BY THE INTRAPERITONEAL ADMINISTRATION OF FRESH ANTERIOR HYPOPHYSEAL SUBSTANCE*

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Communicated by L. Hektoen, December 23, 1921

At the last session (Proc. Amer. Assoc. Anat., *Anat. Rec.*, vol. 21) we reported a characteristic acceleration of growth in rats treated intraperitoneally with the finely ground, fresh anterior lobe of the hypophysis of beef. We had shown that this effect is in marked contrast to the lack of effects from oral administration of the same substance, even in very large amounts. Two careful series of experiments with litter mate controls have now been completed, some members of the first series having been experimented with continuously for a period of over one year. *Such animals are invariably much heavier than their litter mate sisters.* The greatest disparity which has been observed was attained on the 333rd day of life when an animal receiving anterior hypophyseal substance weighed 596 grams and its healthy litter mate control weighed 248 grams. It would not appear to be incorrect to characterize these changes as producing constantly a certain degree of true gigantism. Increase in weight results to a great extent from a storage of fat, but is not solely due to this, the skeleton being invariably somewhat larger and heavier, and, as would be expected, the heart, lung, alimentary canal and kidney are heavier. The fat deposits in the omentum and mesentery are such that these structures weigh three or four times as much in the experimental animals as in their controls. The weight of the hypophysis, thyroid and thymus is not appreciably effected.

Most surprising is the effect on the reproductive system. *Oestrus, as detected by typical changes in the vaginal smear, may never occur in these animals or may be exhibited only at long intervals.* It was, consequently, surprising to find that in all instances the ovaries instead of being underdeveloped weighed twice as much as they did in the control animals and exhibited great numbers of substantial corpora lutea. The uterus, on the contrary, weighed absolutely about half as much as it did in the normal